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FAST TRACK COMMUNICATION

New formulation of nonlinear vector coherent states of f-deformed spin–orbit Hamiltonians

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Abstract
We provide, with a new formulation of vector, coherent states for nonlinear spin–orbit Hamiltonian models in terms of the matrix eigenvalue problem for generalized annihilation operators. Nonlinear quaternion vector coherent states are also discussed.

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Nonlinear coherent states [1] and their generalization as nonlinear vector coherent states (NVCSs) became the focus of attention of research activities for their relevance in nonlinear quantum optics [2–5]. In previous works [4, 5], it has been shown that classes of generalized spin–orbit Hamiltonians (encompassing in specific parameter limits well-known Hamiltonians such as the Jaynes-Cummings Hamiltonian [6] in quantum optics, Rashba [7] and Dresselhaus [8] Hamiltonians in condensed matter physics) associated with f-deformed oscillator algebra [9] remain exactly solvable in the rotating wave approximation. Moreover, classes of NVCSs investigated for the same models prove to be well defined fulfilling all axioms of the Gazeau–Klauder scheme [10]. The vector character of those NVCSs is described in terms of unit sphere S_2 vectors.

In this communication, we propose an alternative construction of Gazeau–Klauder NVCSs for classes of Hamiltonians with spin–orbit interaction entailing the definition of matrix vector coherent states. The specific instance of quaternion NVCSs is also discussed.

Consider the generalized spin–orbit potential parameterized by the couple $(k, \varepsilon) \in (\mathbb{N}/\{0\}) \times \{\pm\}$,

$$V_{k,\varepsilon} = B^{+}_{k,\varepsilon}\sigma_{+} + B^{-}_{k,\varepsilon}\sigma_{-} \qquad B^{+}_{k,\varepsilon} = A^{-\varepsilon k}\lambda^{\varepsilon}(N) \qquad B^{-}_{k,\varepsilon} = \lambda^{-\varepsilon}(N)A^{\varepsilon k}, \tag{1}$$

where, given the Pauli-spin matrices $(\sigma_1, \sigma_2, \sigma_3)$, one has $\sigma_{\pm} = (\sigma_1 \pm \mathrm{i}\sigma_2)/2$, while, introducing $\{a^- := a, a^+ := a^\dagger, N\}$ generators of the ordinary Fock–Heisenberg algebra, the nonlinear operators $A^- := a^- f(N) := af(N)$, $A^+ := f(N)a^+ := f(N)a^\dagger$ and $\{N\} := A^+A^-$ span the Jannussis *et al* f-deformed oscillator algebra [9] assuming that $f(N)$ is a real-valued and non-vanishing operator. Then, we define for $k \in \mathbb{N}/\{0\}$, $\varepsilon = \pm$, $A^{\varepsilon k} := (A^\varepsilon)^k$. The complex-valued operator $\lambda^+(N) := \lambda(N)$ corresponds to the so-called intensity dependent (ID) coupling [11] whereas $\lambda^-(N)$ denotes its complex conjugate. The potential $V_{k,\varepsilon}$ describes a nonlinear model involving a k-multiphoton contribution to the ID coupling $\lambda(N)$ which is of interest in the study of ID interaction between a single atom and the radiation field with the atom making k-photon transitions in QO [11, 12] as well as in the study of the quantized motion of a single ion in an anharmonic oscillator potential trap [13].

The $(k, \varepsilon, \kappa, f)$-deformed spin–orbit model is introduced by the reduced (dimensionless) Hamiltonian [5]

$$\mathcal{H}^{\text{red}}_{k,\varepsilon} = \frac{\hbar\omega}{2\hbar\omega_0}(\{N+1\}+\{N\}) + \frac{1}{2}(\{N+1\}-\kappa\{N\})\sigma_3 + B^+_{k,\varepsilon}\sigma_+ + B^-_{k,\varepsilon}\sigma_- \tag{2}$$

with κ being a new real parameter, ω the radiation field mode frequency and ω_0 the atomic frequency. Now, let us consider $\omega = (1+\epsilon)\omega_0$, assuming that $\omega_0 \neq 0$. This condition is related to the rotating wave approximation (rwa) if the detuning parameter $|\epsilon| \ll 1$ and provided $|\omega - \omega_0| \ll \omega, \omega_0$. Nonetheless, if ω_0 turns to be negative, then the relation $\omega = (1+\epsilon)\omega_0$ implies that $\epsilon < -1$ so that we are far from the rwa regime.

Hamiltonian (2) generalizes known Hamiltonians appearing in the literature, including linear and nonlinear spin–orbit effects recovering for particular parameters: $(k, \varepsilon, \kappa, \lambda(N), f(N)) = (1, +, 1, \text{c}, 1)$ Hussin and Nieto [14]; $(k, \varepsilon, \kappa, \lambda(N), f(N)) = (1, -, 1, i\text{c}, 1)$ Shen *et al* [15]; $(k, \varepsilon, \kappa, \lambda(N), f(N)) = (1, +, q, \text{c}, [(1-q^N)/(N(1-q))]^{1/2})(p=1, q)$ Ben Geloun *et al* [4]; $(k, \varepsilon, \kappa, \lambda(N), \epsilon) = (1, +, 1, \text{c}, 0)$ Balantekin *et al* [16]; $(k, \varepsilon, \kappa, \lambda(N), f(N)) = (1, +, 1, c, N+m)$ Daoud and Douary [13], c being some real constant in these models.

Let us consider the spectrum of Hamiltonian (2). It can be shown that this Hamiltonian is exactly solvable in the Hilbert space of states $\mathcal{V}$ spanned by the usual Fock representation space $\{|n\rangle, n = 0, 1, \ldots\}$ tensorized by the spin eigenstates of σ_3, i.e., $\sigma_3|\pm\rangle = \pm|\pm\rangle$. We first obtain a finite sequence of eigenstates $\{|E^*_q\rangle, q = 0, 1, \ldots, k-1\}$, such that

$$|E^*_q\rangle = |q, -\varepsilon\rangle \tag{3}$$

with associated eigenvalues

$$E^*_q = \tfrac{1}{2}[(1+\epsilon-\varepsilon)\{q+1\} + (1+\epsilon+\varepsilon\kappa)\{q\}]. \tag{4}$$

Formally, the ground state and its energy may be expressed as $|E^*_0\rangle$ and E^*_0.

Now, let us introduce the quantities, for any $(n+k\varepsilon) \in \mathbb{N}$,

$$\begin{aligned}\mathcal{E}(\{n\}) &= \frac{1}{2}\left[\frac{\epsilon}{2}\{n+k\varepsilon+1\} + \frac{1}{2}(1+\epsilon+\kappa)\{n+k\varepsilon\} - \left(1+\frac{\epsilon}{2}\right)\{n+1\} - \frac{1}{2}(1+\epsilon-\kappa)\{n\}\right] \\ Q(\{n\}) &= \left[\mathcal{E}^2(\{n\}) + |\lambda(n+k\varepsilon)|^2\left(\frac{\{n+k\varepsilon\}!}{\{n\}!}\right)^{\varepsilon}\right]^{\frac{1}{2}}\end{aligned} \tag{5}$$

where the notation $\{n\} := nf^2(n)$ is referred to as the f-basic number. For any $n + k\varepsilon \geqslant 0$, the eigenenergies are given by

$$\begin{aligned}E^\pm_n = \frac{1}{2}\bigg[&\frac{\epsilon}{2}\{n+k\varepsilon+1\} + \frac{1}{2}(1+\epsilon+\kappa)\{n+k\varepsilon\} + \left(1+\frac{\epsilon}{2}\right)\{n+1\} \\ &+ \frac{1}{2}(1+\epsilon-\kappa)\{n\}\bigg] \pm Q(\{n\}).\end{aligned} \tag{6}$$

Furthermore, from the definition of the mixing angle $\vartheta(\{n\})$ as

$$\sin\vartheta(\{n\}) = \mathrm{e}^{\mathrm{i}\varphi_\lambda(n)}\left[\frac{Q(\{n\}) - \mathcal{E}(\{n\})}{2Q(\{n\})}\right]^{\frac{1}{2}} \qquad \cos\vartheta(\{n\}) = \left[\frac{Q(\{n\}) + \mathcal{E}(\{n\})}{2Q(\{n\})}\right]^{\frac{1}{2}} \tag{7}$$

with $\exp(\mathrm{i}\varphi_\lambda(n)) = \lambda(n+k\varepsilon)/|\lambda(n+k\varepsilon)|$ the phase factor of $\lambda(n+k\varepsilon)$, the eigenstates associated with the eigenvalues (6) can be written as

$$\left|E_n^+\right\rangle = \sin\vartheta(\{n\})|n,+\rangle + \cos\vartheta(\{n\})|n+k\varepsilon,-\rangle \tag{8}$$

$$\left|E_n^-\right\rangle = \cos\vartheta(\{n\})|n,+\rangle - \overline{\sin\vartheta(\{n\})}|n+k\varepsilon,-\rangle, \tag{9}$$

with $n \geqslant k$ for $\varepsilon = -$ and the notation $\overline{X}$ denoting the complex conjugate of the quantity X. Henceforth, we set $n_0^\varepsilon := \max(0, -k\varepsilon)$ and $\widetilde{n} = n + n_0^\varepsilon$.

The total Hilbert space $\mathcal{V}$ admits the spectral decomposition which consists of a direct sum of one-dimensional subspaces $\mathcal{V}_q$ separately spanned by the states $|E_q^*\rangle$, $q = 0, 1, \ldots, k-1$, with the complementary $\overline{\mathcal{V}}$ of $\oplus_{q=0}^{k-1}\mathcal{V}_q$ in $\mathcal{V}$, generated by two towers of energy eigenstates $\left\{\left|E_{\widetilde{n}}^\pm\right\rangle, \widetilde{n} = n + n_0^\varepsilon, n = 0, 1, 2, \ldots\right\}$, namely $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_{k-1} \oplus \overline{\mathcal{V}}$.

A strictly increasing and positive energy spectrum is generally required for Gazeau–Klauder coherent state construction. However, we do not impose those severe conditions on the spectrum. The minimal requirement allowing the NVCS building that we will refer to is that of a bounded from below spectrum $E_{\widetilde{n}}^\pm > E_{n_0^\varepsilon}^\pm$. Note that in [14] (see also references therein), a good 'detuning' of the parameters reveals to be efficient in order to obtain an increasing and positive spectrum. Through an argument of continuity, the minimal requirement $E_{\widetilde{n}}^\pm > E_{n_0^\varepsilon}^\pm$ may be satisfied regarding the freedom afforded by the parameters $(k, \varepsilon, \kappa, \epsilon, \lambda(N))$ together with the deformation operator $f(N)$.

The following operators provide the passage from one basis of $\mathcal{V}$ to another,

$$\mathcal{U} = \sum_{n=0,\pm}^{\infty}\left|E_{\widetilde{n}}^\pm\right\rangle\langle n,\pm| \qquad \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty}|n,\pm\rangle\left\langle E_{\widetilde{n}}^\pm\right|, \tag{10}$$

and are mutually adjoint on $\overline{\mathcal{V}}$ even though nonunitary on $\mathcal{V}$. However, we have $\mathcal{U}^\dagger\mathcal{U} = \mathbb{I}_{\mathcal{V}}$ and $\mathcal{U}\mathcal{U}^\dagger = \mathbb{I}_{\overline{\mathcal{V}}}$.

Given the complex-valued quantities $K_\pm(\{n\})$, ladder operators on $\mathcal{V}$ can be defined such that

$$\mathcal{A}^- = \sum_{n=0,\pm}^{\infty}\left|E_{\widetilde{n}-1}^\pm\right\rangle K_\pm(\{n\})\left\langle E_{\widetilde{n}}^\pm\right| \qquad \mathcal{A}^+ = \sum_{n=0,\pm}^{\infty}\left|E_{\widetilde{n}+1}^\pm\right\rangle\overline{K_\pm(\{n+1\})}\left\langle E_{\widetilde{n}}^\pm\right| \tag{11}$$

and prove to be mutually adjoint on the subspace $\overline{\mathcal{V}}$. Factoring out $\mathcal{U}$ and $\mathcal{U}^\dagger$ in (11), it is easy to get the diagonal operators $\mathbb{A}^\pm = \mathcal{U}^\dagger\mathcal{A}^\pm\mathcal{U}$ defined on the basis $|n,\pm\rangle$,

$$\mathbb{A}^- = \sum_{n=0,\pm}^{\infty}K(\{n\})|n-1,\pm\rangle\langle n,\pm| \qquad \mathbb{A}^+ = \sum_{n=0,\pm}^{\infty}\overline{K(\{n+1\})}|n+1,\pm\rangle\langle n,\pm| \tag{12}$$

$K(\{n\}) := \mathrm{diag}(K_+(\{n\}), K_-(\{n\}))$ is a 2×2 diagonal complex matrix and $\overline{K(\{n\})} := \mathrm{diag}(\overline{K_+(\{n\})}, \overline{K_-(\{n\})})$.

We introduce the NVCSs by the generalized eigenvalue problem

$$\mathcal{A}^-|(z,w);\tau_\pm;\pm\rangle = \widetilde{\mathfrak{Z}}(z,w)\widetilde{\mathbb{Q}}_{\mathcal{V}}|(z,w);\tau_\pm;\pm\rangle \tag{13}$$

where $\widetilde{\mathfrak{Z}}$ is a 2×2 matrix operator depending on two complex variables (z,w), $\tau_\pm$ are real parameters introduced in the following, $\pm$ correspond to the spin-vector dependence replacing the S_2 unit sphere-vector dependence in the formulation of [4, 5]. Defined on $\mathcal{V}$, the operator $\widetilde{\mathbb{Q}}_{\mathcal{V}}$ is given by

$$\widetilde{\mathbb{Q}}_{\mathcal{V}} = \sum_{q=0}^{k-1} |E_q^*\rangle\langle E_q^*| + \sum_{n=0,\pm}^{\infty} \left|E_{\tilde{n}}^{\pm}\right\rangle h_f^{\pm}(n)\left\langle E_{\tilde{n}}^{\pm}\right|, \tag{14}$$

and the quantities $h_f^{\pm}(n) \neq 0$ are complex scalars such that $h_f^{\pm}(n) \to 1$ as $f(N) \to 1$. Assume that there exists $\mathfrak{Z} = \mathcal{U}^{\dagger}\widetilde{\mathfrak{Z}}\mathcal{U}$ a complex constant matrix and $\mathbb{Q}_{\mathcal{V}} = \mathcal{U}^{\dagger}\widetilde{\mathbb{Q}}_{\mathcal{V}}\mathcal{U}$, such that we translate (13) into the diagonal basis as

$$\begin{aligned} &\mathbb{A}^{-}|\mathfrak{Z};\tau_{\pm};\pm\rangle = \mathfrak{Z}\mathbb{Q}_{\mathcal{V}}|\mathfrak{Z};\tau_{\pm};\pm\rangle && |\mathfrak{Z};\tau_{\pm};\pm\rangle = \mathcal{U}^{\dagger}|(z,w);\tau_{\pm};\pm\rangle \\ &\mathbb{Q}_{\mathcal{V}} = \sum_{n=0,\pm}^{\infty} |n\rangle\langle n| \otimes h_f(n)|\pm\rangle\langle\pm| && h_f(n) = \mathrm{diag}\big(h_f^{+}(n), h_f^{-}(n)\big). \end{aligned} \tag{15}$$

Such a matrix eigenvalue problem can be obtained if we fix $\mathfrak{Z} = \mathrm{diag}(z, w), z, w \in \mathbb{C}$. A second example of NVCSs defined over matrix domain is realized considering $\mathfrak{Z}$ as a quaternionic matrix.

Let us then set $\mathfrak{Z} = \mathrm{diag}(z, w), z, w \in \mathbb{C}$ with, consequently,

$$\widetilde{\mathfrak{Z}}(z,w) = \sum_{n=0}^{\infty} \left(\left|E_{\tilde{n}}^{+}\right\rangle z\left\langle E_{\tilde{n}}^{+}\right| + \left|E_{\tilde{n}}^{-}\right\rangle w\left\langle E_{\tilde{n}}^{-}\right|\right).$$

The treatment of the eigenvalue problem (15) is straightforward by expanding $|\mathfrak{Z};\tau_{\pm};\pm\rangle$ as a power series of $|n\rangle \otimes |\pm\rangle$ with matrix valued complex continuous functions $C_n(\mathfrak{Z}) = \mathrm{diag}\big(C_n^{+}(\mathfrak{Z}), C_n^{-}(\mathfrak{Z})\big)$. One gets the matrix recurrence relation

$$\forall n \in \mathbb{N} \qquad C_{n+1}(\mathfrak{Z})K(\{n+1\}) = \mathfrak{Z}h_f(n)C_n(\mathfrak{Z}) \tag{16}$$

easily solved by, setting the initial condition $K_{\pm}(\{0\}) = 0$,

$$C_n(\mathfrak{Z}) = R(n)C_0(\mathfrak{Z})\mathfrak{Z}^n \qquad R(n) = (K(\{n\})!)^{-1}(h_f(n-1)!h_f(0)) \tag{17}$$

where $K(\{n\})! := \prod_{k=1}^{n} K(\{k\})$ and $K(\{0\})! := \mathbb{I}_2$, $h_f(n)! := \prod_{k=1}^{n} h_f(k)$, $h_f(0)! := \mathbb{I}_2$ and $h_f(-1)! := (h_f(0))^{-1}$, by convention. The constant matrix $C_0(\mathfrak{Z})$ is yet to be determined.

The stability of NVCSs under a time evolution operator $U(t) = \exp\left[-\mathrm{i}\omega_0 t\mathbb{H}_{k,\varepsilon}^{\mathrm{red}}\right]$, with $\mathbb{H}_{k,\varepsilon}^{\mathrm{red}} = \mathcal{U}^{\dagger}\mathcal{H}_{k,\varepsilon}^{\mathrm{red}}\mathcal{U}$, is now studied. Consider $K_{\pm}(\{n\}) = \exp(\mathrm{i}\varphi_{\pm}(\{n\}))K_{\pm}^{0}(\{n\})$, with $K_{\pm}^{0}(\{n\})$ real quantities, i.e. in terms of matrices, $K(\{n\}) = \exp(\mathrm{i}\varphi(\{n\}))K^{0}(\{n\})$, with appropriate diagonal matrices of phase $\varphi(\{n\})$ and norm $K^{0}(\{n\})$ entries. Then, one requires $\varphi(\{n\}) = \omega_0\tau(E_{\tilde{n}} - E_{\widetilde{n-1}})$, where $\tau = \mathrm{diag}(\tau_+, \tau_-)$ and $E_{\tilde{n}} = \mathrm{diag}\big(E_{\tilde{n}}^{+}, E_{\tilde{n}}^{-}\big)$, such that $U(t)|\mathfrak{Z};\tau_{\pm};\pm\rangle = |\mathfrak{Z};\tau_{\pm}+t;\pm\rangle$.

The normalizability of NVCSs is governed by the trace class scalar product

$$\sum_{\pm}\langle\mathfrak{Z};\tau_{\pm};\pm|\mathfrak{Z};\tau_{\pm};\pm\rangle = 1. \tag{18}$$

Let us introduce the quantities

$$C_0(\mathfrak{Z}) = N(\mathfrak{Z})\exp[-\mathrm{i}\omega_0\tau E_{\tilde{n}}] \qquad N(\mathfrak{Z})^{-2} = \sum_{n=0}^{\infty}\left(|z|^{2n}\left(R_{+}^{0}(n)\right)^2 + |w|^{2n}(R_{-}^{0}(n))^2\right) \tag{19}$$

where $R^0(n) = \mathrm{diag}\big(R_{+}^{0}(n), R_{-}^{0}(n)\big) = (K^0(\{n\})!)^{-1}(h_f(n-1)!h_f(0))$. Note that the convergence radii of the series (19) are such that $|z| \leqslant L_+, |w| \leqslant L_-$ and $L_{\pm} = \lim_{n\to\infty} K_{\pm}^{0}(\{n\})\big/h_f^{\pm}(n-1)$. The general set of NVCSs, solution of the eigenvalue problem (15), stable under time translations and normalized to unity in the sense of (19), are given by

$$|\mathfrak{Z};\tau_{\pm};\pm\rangle = N(\mathfrak{Z})\sum_{n=0}^{\infty}|n\rangle \otimes R_0(n)\exp[-\mathrm{i}\omega_0\tau E_{\tilde{n}}]\mathfrak{Z}^n|\pm\rangle. \tag{20}$$

It is noteworthy that defining NVCSs by (13), or equivalently mapping (20) on the basis $\left|E^{\pm}_{\tilde{n}}\right\rangle$, implies that the eigenvalue $\tilde{\mathfrak{z}} = 0$ is associated with any combination of the states $|E^{*}_{q}\rangle$ and $\left|E^{\pm}_{n_0^{\varepsilon}}\right\rangle$. This fact breaks the continuity of NVCSs in the vicinity of $\tilde{\mathfrak{z}} = 0$. However, we remove this singularity by defining the NVCSs associated with $\tilde{\mathfrak{z}} = 0$ merely as a combination of $\left|E^{\pm}_{n_0^{\varepsilon}}\right\rangle$.

The overcompleteness property of NVCSs consists in the existence of the resolution of the identity on $\mathcal{V}$, namely

$$\sum_{\pm}\int_{D} \mathrm{d}\mu(\mathfrak{z})|\mathfrak{z};\tau_{\pm};\pm\rangle\langle\mathfrak{z};\tau_{\pm};\pm| = \mathbb{I}_{\mathcal{V}}. \tag{21}$$

Using polar coordinates $z = r_{+}\mathrm{e}^{\mathrm{i}\theta_{+}}$ and $w = r_{-}\mathrm{e}^{\mathrm{i}\theta_{-}}$, $r_{\pm} \geqslant 0$, $\theta_{\pm} \in [0, 2\pi[$, the measure $\mathrm{d}\mu(\mathfrak{z})$ is parameterized by

$$\mathrm{d}\mu(\mathfrak{z}) = \mathcal{W}_{+}(r_{+})\mathcal{W}_{-}(r_{-})r_{+}r_{-}\mathrm{d}r_{+}\mathrm{d}r_{-}\mathrm{d}\theta_{+}\mathrm{d}\theta_{-} \tag{22}$$

with weight functions $\mathcal{W}_{\pm}(r_{\pm})$ to be specified. A direct calculation leads to the Stieljes moment problems

$$\int_{0}^{L_{\pm}^{2}} \mathrm{d}u_{\pm}u_{\pm}^{n}h_{\pm}(u_{\pm}) = \left(K_{\pm}^{0}(\{n\})!\right)^{2}/\left(h_{f}^{\pm}(n-1)!h_{f}^{\pm}(0)\right)^{2} \tag{23}$$

with $u_{\pm} = r_{\pm}^{2}$ and $h_{\pm}(u_{\pm}) = \pi\mathcal{W}_{\pm}(u_{\pm})(N(\mathfrak{z}))^{2}$. Constraining the ladder operator algebra such that $[\mathcal{A}^{-},\mathcal{A}^{+}] = \{N+1\} - \{N\}$ imposes the choice $K^{0}(\{n\}) = \sqrt{\{n\}}\mathbb{I}_{2}$. One removes the deformed part in $f(n)$ by letting $h_{f}^{\pm}(n) = s|f(n+1)|$, $n \geqslant 0$, $s = \pm 1$. Noting that $L_{\pm} = \infty$, solutions to (23) are readily obtained as $h_{\pm}(u_{\pm}) = \mathrm{e}^{-u_{\pm}}$ and one deduces the corresponding weight factors. The overall measure is then

$$\mathrm{d}\mu(\mathfrak{z}) = \frac{1}{\pi^{2}}\left(\mathrm{e}^{-r_{+}^{2}} + \mathrm{e}^{-r_{-}^{2}}\right)r_{+}r_{-}\mathrm{d}r_{+}\mathrm{d}r_{-}\mathrm{d}\theta_{+}\,\mathrm{d}\theta_{-}. \tag{24}$$

A similar result is worked out for the supersymmetric harmonic oscillator VCSs [17]. We can deduce the measure corresponding to the NCVSs $|\tilde{\mathfrak{z}};\tau_{\pm};\pm\rangle$ by setting $\mathrm{d}\mu(\tilde{\mathfrak{z}}) = \mathrm{d}\mu(\mathfrak{z})$.

As particular NVCSs defined over the matrix domain, let us consider now quaternionic NVCSs recovered for the following matrix representation [2, 17],

$$\mathfrak{z}_{\mathrm{quat}} = r(\cos\xi\,\mathbb{I}_{2} + \mathrm{i}\sin\xi\,\hat{\sigma}) \qquad \hat{\sigma} = \begin{pmatrix} \cos\theta & \mathrm{e}^{\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{-\mathrm{i}\phi}\sin\theta & -\cos\theta \end{pmatrix}, \tag{25}$$

where, in the polar coordinates, $z = r\mathrm{e}^{\mathrm{i}\xi}$ with $r > 0$, $\xi \in [0, 2\pi[$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi[$, the corresponding $\tilde{\mathfrak{z}}(z, w)$ can be easily found. Then, quaternionic NVCSs associated with spin–orbit Hamiltonians can be written in the basis $|n, \pm\rangle$

$$\begin{aligned} |\mathfrak{z}_{\mathrm{quat}};\tau_{\pm};\pm\rangle &= N(\mathfrak{z}_{\mathrm{quat}})\sum_{n=0}^{\infty}|n\rangle\otimes R_{0}(n)\exp[-\mathrm{i}\omega_{0}\tau E_{\tilde{n}}]\mathfrak{z}_{\mathrm{quat}}^{n}|\pm\rangle \\ (N(\mathfrak{z}_{\mathrm{quat}}))^{-2} &= \sum_{n=0}^{\infty}|z|^{2n}\left(\left(R_{+}^{0}(n)\right)^{2} + \left(R_{-}^{0}(n)\right)^{2}\right) \end{aligned} \tag{26}$$

the normalization factor being only defined on the disc D of radius

$$L = \lim_{n\to\infty}\left[\frac{\left(R_{+}^{0}(n+1)\right)^{2} + \left(R_{-}^{0}(n+1)\right)^{2}}{\left(R_{+}^{0}(n)\right)^{2} + \left(R_{-}^{0}(n)\right)^{2}}\right]^{-1/2}. \tag{27}$$

The resolution of the identity over $\mathcal{V}$ with respect to (26) can be considered through (21) with adapted considerations for states, the integration domain $\mathcal{D} = D \times S_2$ and using the parametrization of the measure

$$\mathrm{d}\mu(\mathfrak{Z}_{\mathrm{quat}}) = \mathcal{W}(r) r \, \mathrm{d}r \, \mathrm{d}\xi \, \mathrm{d}\mu_{S_2}(\theta, \phi) \qquad \mathrm{d}\mu_{S_2}(\theta, \phi) = \frac{1}{4\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{28}$$

After some algebra and changing the variables as $u = r^2$, one comes to the moment problems

$$\int_0^{L^2} \mathrm{d}u \, u^n h(u) = \left(K^0_{\pm}(\{n\})!\right)^2 / \left(h^{\pm}_f(n-1)! h^{\pm}_f(0)\right)^2, \tag{29}$$

where $h(u) = \pi \mathcal{W}(u)(N(\mathfrak{Z}_{\mathrm{quat}}))^2$. The further condition $R^0_+(n) = R^0_-(n)$ should be imposed in order to obtain a unique moment identity which, by constraining both ladder operator algebra and $h_f(n)$ as previously performed, leads to an infinite convergence radius $L = \infty$ for $(N(\mathfrak{Z}_{\mathrm{quat}}))^{-2} = 2\mathrm{e}^u$ and to the solution of (29) as $h(u) = \mathrm{e}^{-u}$. Noting that $\mathcal{W}(r) = 2/\pi$, the measure (28) has the form

$$\mathrm{d}\mu(\mathfrak{Z}_{\mathrm{quat}}) = \frac{1}{2\pi^2} r \, \mathrm{d}r \, \sin\theta \, \mathrm{d}\xi \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{30}$$

Finally, for any type of NVCSs, the identity action axiom [10] can be achieved by giving the action-angle conjugate coordinates $\left(J_{\pm} = \langle \mathfrak{Z}; \tau_{\pm}; \pm | \mathbb{H}^{\mathrm{red}}_{k,\varepsilon} | \mathfrak{Z}; \tau_{\pm}; \pm \rangle, \tau_{\pm}\right)$.

An interesting application concerns the classes of NVCSs for $(p, q, \alpha, \beta = 0, \ell)$-Burban deformed oscillator algebras defined in [18] in relation to bibasic hypergeometric functions [19]

$$f(N) = \sqrt{\frac{p^{-\alpha N} - q^{\alpha N}}{N(p^{-\ell} - q^{\ell})}}$$

where the real parameters $p > 1, 0 < q < 1$ are such that $(pq)^{\alpha} < 1, \alpha > 0, \ell \in \mathbb{R}$; the basic integer number is denoted by $[n]$.

Consider now the $(k, \varepsilon, \kappa, f)$-model under this deformation. It appears also possible to uniquely determine the NVCSs in this situation once one imposes the heretofore conditions on the ladder operator algebra and the matrix $h_f(n)$.

However, let us proceed differently by considering the following construction leading to a new set of solvable NVCSs. Let us ask for the ladder operators $\mathbb{A}^{\pm}$ to satisfy a $(p, q, \alpha, 0, \ell)$-deformed algebra on $\mathcal{V}$ such that

$$\begin{aligned} \mathbb{A}^- \mathbb{A}^+ - q^{\ell} \mathbb{A}^+ \mathbb{A}^- &= p^{-\alpha N} = \sum_{n=0,\pm}^{\infty} |n, \pm\rangle p^{-\alpha n} \langle n, \pm| \\ \mathbb{A}^- \mathbb{A}^+ - p^{-\ell} \mathbb{A}^+ \mathbb{A}^- &= q^{\alpha N} = \sum_{n=0,\pm}^{\infty} |n, \pm\rangle q^{\alpha n} \langle n, \pm|. \end{aligned} \tag{31}$$

Then, by induction, the ladder operators $\mathcal{A}^{\pm} = \mathcal{U} \mathbb{A}^{\pm} \mathcal{U}^{\dagger}$ acting on the subspace $\overline{\mathcal{V}}$ are also constrained to obey the same kind of algebras. Expanding (31), the following recurrence relations are obtained:

$$\left(K^0_{\pm}([n+1])\right)^2 - q^{\ell}\left(K^0_{\pm}([n])\right)^2 = p^{-\alpha n}, \qquad \left(K^0_{\pm}([n+1])\right)^2 - p^{-\ell}\left(K^0_{\pm}([n])\right)^2 = q^{\alpha n}. \tag{32}$$

Given the initial values $K^0_{\pm}([0]) = 0$, the solutions to the associated recurrence relations (32) are given by $K^0([n]) = \sqrt{[n]} \mathbb{I}_2$.

We now distinguish the NVCSs defined for $\mathfrak{Z} = \mathrm{diag}(z, w)$, i.e. $\mathfrak{Z}$-NVCSs (resp. for the quaternion matrix $\mathfrak{Z}_{\text{quat}}$ named $\mathfrak{Z}_{\text{quat}}$-NVCSs), from the above NVCSs by setting the matrix

$$\begin{aligned} &h_f(N) = h_0(N) = q^{\mu N}\mathrm{diag}(\sqrt{l^+(p,q)}, \sqrt{l^-(p,q)}) \\ &(\text{resp. } h_f(N) = h_{\text{quat}}(N) = q^{\mu N}\sqrt{l(p,q)}\mathbb{I}_2) \end{aligned} \tag{33}$$

where μ is another real deformation parameter. The positive real-valued functions $l^{\pm}(p, q)$ (resp. $l(p, q)$) are such that the general requirement $\lim_{(p,q)\to(1^+,1^-)} h_0(N) = \mathbb{I}_2$ (resp. $\lim_{(p,q)\to(1^+,1^-)} h_{\text{quat}}(N) = \mathbb{I}_2$) holds. It turns out that both NVCSs are well defined with solvable moment problems associated with the resolution of identity on $\mathcal{V}$ if $\mu = \alpha/2$.

The normalization factor for $\mathfrak{Z}$- (resp. $\mathfrak{Z}_{\text{quat}}$-) NVCSs is then defined by the series

$$\begin{aligned} (\mathcal{N}(\mathfrak{Z}))^{-2} &= \sum_{n=0}^{\infty}(q^{\alpha/2})^{n(n-1)}\left(\frac{|z|^{2n}}{[n]!}(l^+(p,q))^n + \frac{|w|^{2n}}{[n]!}(l^-(p,q))^n\right) \\ &= \sum_{\pm}\mathcal{E}^{(1/2,0)}_{(p^\alpha,q^\alpha)}\big(r_\pm^2 q^{-\alpha/2} l^{\pm}(p,q)(p^{-\ell}-q^{\ell})\big) \qquad |z| = r_+ \quad |w| = r_- \end{aligned} \tag{34}$$

$$\begin{aligned} \Bigg(\text{resp. } (\mathcal{N}(\mathfrak{Z}_{\text{quat}}))^{-2} &= 2\sum_{n=0}^{\infty}(q^{\alpha/2})^{n(n-1)}\left(\frac{|z|^{2n}}{[n]!}(l(p,q))^n\right) \\ &= 2\mathcal{E}^{(1/2,0)}_{(p^\alpha,q^\alpha)}(r^2 q^{-\alpha/2} l(p,q)(p^{-\ell}-q^{\ell})) \qquad |z| = r\Bigg) \end{aligned} \tag{35}$$

of infinite convergence radii since the generalized exponential

$$\mathcal{E}^{(\rho,\nu)}_{(\bar{p},\bar{q})}(z) = \sum_{n=0}^{\infty}\left(\frac{\bar{q}^{\rho}}{\bar{p}^{\nu}}\right)^{n^2}\frac{z^n}{[\bar{p},\bar{q};\bar{p},\bar{q}]_n} \tag{36}$$

converges everywhere under the condition $\bar{q}^{2\rho}\bar{p}^{1-2\nu} \leqslant 1$ [4] satisfied for $\rho = 1/2$, $\nu = 0$, $\bar{q} = q^\alpha$ and $\bar{p} = p^\alpha$ leading to $(qp)^\alpha \leqslant 1$.

The moment problems (23) (resp. (29)) then become, for all $n \in \mathbb{N}$,

$$\int_0^{\infty} \mathrm{d}u_\pm u_\pm^n h^{\pm}(u_\pm) = q^{-\alpha n(n-1)/2}(l^{\pm}(p,q))^{-n}([n]!) \tag{37}$$

$$\left(\text{resp. } \int_0^{\infty}\mathrm{d}u u^n h(u) = q^{-\alpha n(n-1)/2}(l(p,q))^{-n}([n]!)\right). \tag{38}$$

Solutions to (37) (resp. to (38)) based on the (p, q)-analogue of the Ramanujan integral [4] can be written as

$$h^{\pm}(r_\pm^2) = \frac{(p^{-\ell}-q^{\ell})l^{\pm}(p,q)}{q^\alpha\log(1/(pq)^\alpha)}e_{(p^\alpha,q^\alpha)}\big(-r_\pm^2 p^{-\alpha/2}q^{-\alpha}l^{\pm}(p,q)(p^{-\ell}-q^{\ell})\big) \tag{39}$$

$$\left(\text{resp. } h(r^2) = \frac{(p^{-\ell}-q^{\ell})l(p,q)}{q^\alpha\log(1/(pq)^\alpha)}e_{(p^\alpha,q^\alpha)}(-r^2 p^{-\alpha/2}q^{-\alpha}l(p,q)(p^{-\ell}-q^{\ell}))\right) \tag{40}$$

where we have introduced the reduction of (36), through $\rho = 0$ and $\nu = 1/2$, as

$$e_{(\bar{p},\bar{q})}(z) = \sum_{n=0}^{\infty}\frac{1}{\bar{p}^{n^2/2}}\frac{z^n}{[\bar{p},\bar{q};\bar{p},\bar{q}]_n} \qquad |z| < (\bar{p})^{-1/2}. \tag{41}$$

The weight functions $\mathcal{W}^{\pm}(r_\pm)$ (resp. $\mathcal{W}(r)$) can be easily deduced from (39) (resp. (40))

$$\mathcal{W}^{\pm}(r_{\pm}) = \frac{1}{\pi}|N(\mathfrak{z})|^{-2}h^{\pm}\left(r_{\pm}^{2}\right) \tag{42}$$

$$\left(\text{resp. } \mathcal{W}(r) = \frac{1}{\pi}|N(\mathfrak{z}_{\text{quat}})|^{-2}h(r^{2})\right). \tag{43}$$

These weight factors differ from the measure weight factors obtained in [4, 5]. Moreover, the solutions to (37) and (38) may be not unique according to the Carleman criterion [20].

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